A Unified View of Optimal Stopping

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In timing problems under certainty an *r-percent stopping rule* prescribes that the program be stopped when the rate of return from delaying the project falls to the force of interest, balancing benefits and costs of stopping. This paper shows that in the case of uncertainty the program is also stopped once the expected rate of return from delaying the project reaches the risk-adjusted force of interest. This rule has much of the intuition of the rule under certainty, and viewing stopping in this way reveals additional theoretical and practical insights.

Keywords: investment timing, stopping rules, investment under uncertainty, r-percent rule, real options, pure postponement value, quasi-option value, value of information, opportunity cost.

JEL Codes: C61, D92, E22, G12, G13, G31, Q00

Q: How can you delay milk turning sour?

A: Keep it in the cow.[1]

1. Introduction

At any time in an industry certain distinct, irreversible economic actions may be available but not undertaken. The actions may be offensive (e.g., construction, expansion, acquisition, or re-opening) or defensive (e.g., abandonment, contraction, divestiture, or temporary closure). Some would be profitable (would produce a positive discounted cash flow) if implemented at the time, but are instead held "on the shelf" until the optimal time. Early analyses of such stopping problems originated in forestry harvesting decisions (Gane 1968), though delayed action is clearly advisable in a wide array of economic problems.

Treatments of stopping under uncertainty, typically within the framework of "real options," have little in common with treatments of equivalent actions under certainty.[2] Moreover, opportunity cost is as critical and intuitive a concept in the timing of economic actions under uncertainty as under certainty, but is rarely invoked explicitly as a principle of timing under uncertainty.[3] This paper uses the concept of opportunity cost to present a unified view of the optimal time to act. Insights are brought out that have been obscured in conventional treatments of stopping under uncertainty.

[1] Child's alleged response on a science test (http://mayvelous.blogspot.com/2006/08/childrens-science-exam.html).

[2] The lack of a common approach to stopping under certainty and uncertainty is brought out in Malliaris and Brock (1982), McDonald and Siegel (1986), Clarke and Reed (1989, 1990a), Dixit (1992), Dixit and Pindyck (1994), and Bar-Ilan and Strange (1999).

[3] Amram and Kulatilaka (1999), Copeland and Antikarov (2001), and Trigeorgis (1996), three popular texts on investment under uncertainty, do not contain the term opportunity cost in their indexes. Moore (2001) uses the term once, and Brach (2003) uses the term twice.

2. Stopping under Certainty

The irreversible, lumpy economic actions we consider involve an initial decision as well as an optimal plan that specifies outputs in future time periods and possible other choices.[4] To allow for closed-form solutions we initially focus on actions that can be delayed indefinitely at no cost. The intensity of the action and the ensuing optimal production plan generally depend on the time of initial action, t_0, and the future equilibrium price path of outputs and inputs, among other things. The firm need not be a price taker. In forcing an interior solution we assume that these equilibrium price paths, along with the optimal production plan, initially yield a return to waiting that is at least equal to the force of interest.

We begin by modeling an action to create an income flow. Let $Y(t_0)$ be the forward net present value (NPV) received by irreversibly sinking a discrete investment cost $C(t_0) \geq 0$ at time $t_0 \geq 0$ in return for a certain, incremental cash flow from time t_0 onward with time t_0 present value $W(t_0)$. As in compound option analysis, $W(t_0)$ can include the value of subsequent timing options. To emphasize that the irreversible economic action need not entail investment, but only an irreversible decision, we assume that $C(t_0) = 0$, so that $Y(t_0) = W(t_0)$.[5] We also assume that $W(t_0)$ is time-varying and differentiable, and that $W(t_0) > 0$ for at least some non-degenerate interval of time. The discount-factor approach of Dixit et al. (1999), incorporating their assumption that holding costs are zero, shows the current value of the project at time $t \leq t_0$ to be

$$\Pi(t, t_0) = D(t, t_0) W(t_0) , \qquad (1)$$

[4] Bar-Ilan and Strange (1999) have reviewed stopping when investment is incremental, identifying that this is not so much a stopping problem as an investment intensity problem, with intensity at times being zero. To isolate timing from intensity we focus on lumpy investments.

[5] If $C(t_0)$ is positive the following analysis holds with Y replacing W.

where $D(t, t_0) = \exp[-\int_t^{t_0} r(s)ds]$ is a (riskless) discount factor integrated over the short rates of interest. The decision concerns when to initiate the project.

Traditional or "Marshallian" NPV analyses presume that $t_0 = t$ the first time that $W(t) > 0$ (Dixit and Pindyck 1994, 4-5, 145-47). Under this timing rule $\Pi(t,t) = D(t,t)W(t) = W(t)$. Marglin (1963) was among the first to point out that this timing rule is not necessarily optimal, even under certainty. Maximizing the present value of the project by choosing the optimal time of action $\hat{t}_0$, and assuming an interior solution $\hat{t}_0 \in (t,\infty)$, we find from (1) that

$$\Pi_{t_0}(t,\hat{t}_0) = D_{t_0}(t,\hat{t}_0)W(\hat{t}_0) + D(t,\hat{t}_0)W'(\hat{t}_0) = 0. \tag{2}$$

The solution to (2) yields $\hat{t}_0$ and the critical threshold $W(\hat{t}_0)$. The time t market or equilibrium (option) value of the (optimally managed) opportunity is then

$$\Pi(t,\hat{t}_0) = D(t,\hat{t}_0)W(\hat{t}_0). \tag{3}$$

The option premium, the value of waiting, is

$$O(t,\hat{t}_0) \equiv \Pi(t,\hat{t}_0) - W(t). \tag{4}$$

From (3), prior to acting the market value of the opportunity is rising at the rate of discount, as it must in equilibrium;

$$\Pi_t(t,\hat{t}_0) = r(t)\Pi(t,\hat{t}_0). \tag{5}$$

Under certainty, therefore, stopping under certainty is usually seen as deciding at time t = 0 to stop the program at time $\hat{t}_0$. Marginal analysis, a weighing up of costs and benefits of acting at any time t_0, is subsumed within this stopping rule. Equation (2), however, also produces a stopping condition that yields an intertemporal comparison central to our analysis,

$$\frac{W'(\hat{t}_0)}{W(\hat{t}_0)} = -\frac{D_{t_0}(t,\hat{t}_0)}{D(t,\hat{t}_0)} = r(\hat{t}_0). \tag{6}$$

At the optimal stopping time the forward NPV of the project is rising at the short rate of interest. The second-order condition requires that, on an interval around $\hat{t}_0$,

$$r(t_0) \leq \frac{W'(t_0)}{W(t_0)} \text{ for } t_0 < \hat{t}_0 \tag{7}$$

and

$$r(t_0) \geq \frac{W'(t_0)}{W(t_0)} \text{ for } t_0 > \hat{t}_0. \tag{8}$$

Equations (6) to (8) can easily be seen to imply value matching and smooth pasting of the option value to the underlying value in the temporal domain. A simple change of variable shows that, consonant with the usual perspective on stopping under uncertainty, value matching and smooth pasting also hold in the value domain (Cairns and Davis 2006). Equations (6) to (8), therefore, constitute an *r*% rule for stopping under certainty: act only when the rate of return from delaying investment, in this case capital gains equal to the instantaneous rate of rise of forward NPV, falls to the contemporaneous force of interest. While this rule has previously been identified as a timing rule for land and natural resource developments, where it has been called Wicksell's rule (e.g., Clarke and Reed 1988, 1989, 1990a), it is clearly very general, holding for all types projects, including projects with subsequent timing options (compound options), in all types of markets, competitive and non-competitive (Cairns and Davis 2006).

Viewing stopping as an *r*% rule facilitates intertemporal marginal analysis in the stopping decision. At any time t_0 the instantaneous flow of opportunity benefits from immediate action, namely interest on proceeds $r(t_0)W(t_0)$, can be compared with the instantaneous flow opportunity cost of immediate action, namely lost capital appreciation $W'(t_0)$. By analogy to Mensink and Requate's (2005) two-period model of investment timing under uncertainty, we define the difference

$$W'(t_0) - r(t_0)W(t_0) \quad (9)$$

as *pure postponement flow*. Using equations (1) and (4) this can be discounted and integrated over the life of the option to form option premium $O(t,\hat{t}_0)$ identified in (4),

$$O(t,\hat{t}_0) = \int_t^{\hat{t}_0} O_{t_0}(t,t_0)dt_0 = \int_t^{\hat{t}_0} D(t,t_0)\big(W'(t_0) - r(t_0)W(t_0)\big)dt_0 . \quad (10)$$

Maximizing $\Pi(t,t_0)$ in equation (2) is equivalent to maximizing $O(t,t_0)$ in equation (10): it is optimal to strike only when pure postponement flow falls to zero, at which point the flow opportunity cost of stopping, $W'(t_0)$, equals the flow opportunity benefit of stopping, $r(t_0)W(t_0)$. When pure postponement flow is positive the net opportunity cost of stopping is positive and it is optimal to delay (equation 7).[6] When pure postponement flow is negative the optimal stopping time has passed (equation 8).

The intuition from presenting the stopping decision in this light is impeccable. The slower the rate of rise of forward NPV or the higher the opportunity cost of delay, the sooner the program is stopped. Contrary to some interpretations, waiting is not valuable solely because of deferment of a fixed cost of investment; in the derivation of our stopping rule investment cost *C* is zero. Nor is waiting only valuable under uncertainty. It is growth in project NPV at a rate greater than the rate of interest that induces waiting. Given the option to wait, even projects whose value is negative if implemented now can have a positive market value due to an option premium derived from pure postponement flow. In evaluating the timing option using this stopping rule practitioners need not immediately solve the stopping problem for $\hat{t}_0$ or $W(\hat{t}_0)$. Nor do they need to calculate and track the integral option premium $O(t,\hat{t}_0)$, acting when it goes to zero. They need only continuously weigh, in true

[6] With $D(t,t_0) > 0$, Sign$\left\{D(t,t_0)\big(W'(t_0) - r(t_0)W'(t_0)\big)\right\}$ = Sign$\left\{W'(t_0) - r(t_0)W'(t_0)\right\}$.

neoclassical form, opportunity costs and opportunity benefits of stopping at the intertemporal margin, and act when the two are equal.

Figure 1 plots equations (6) to (8) for a range of stopping times for a project with no investment cost but a non-constant rate of growth of forward NPV. In accordance with the *r*% rule the program is optimally stopped when the rate of appreciation of forward NPV falls to *r* from above. The example is based on Wicksell's insights for a simple point-output problem, serving wine, presented in Appendix 1. This example illustrates that the *r*% stopping rule is applicable to any lumpy economic decision – consumption providing utility as well as action providing monetary gain.

3. Stopping under Uncertainty

Stopping under uncertainty is traditionally seen to be "much more complicated" (Brealey and Myers 2003, p. 138). The typical optimal stopping rule under uncertainty is to act as soon as the state variable *W* reaches some endogenously determined trigger or boundary value $\hat{W}$ for the first time (Brock et al. 1989, Dixit and Pindyck 1994):

$$\hat{t}_0 = \inf \left\{ t_0 \middle| W(t_0) = \hat{W} \right\}. \tag{11}$$

The derivation of the stopping trigger is conducted in the value, rather than the time, domain, and as such stopping is no longer explicitly identified as an intertemporal comparison of costs and benefits. Another difference between decisions under certainty and uncertainty is that under certainty the timing *decision* can be made at time *t*, even if the economic *action* may be optimally postponed to $\hat{t}_0 > t$. Under uncertainty the timing decision is continually deferred until specified stopping conditions are satisfied, e.g. equation (11), at which point the decision and the economic action are concurrent.

In this section we derive an *r*% stopping rule that is identical in concept, and remarkably similar in form, to the rule derived under certainty. The rule highlights the fact that opportunity cost and benefits of stopping are continually being reassessed at the intertemporal margin as new information becomes available.

Once again we focus our analysis on situations that yield an interior stopping point. Let the state variable W be described by a density function of which the moments are assumed to be known. To facilitate closed-form solutions we represent changes in value as the one-dimensional, autonomous diffusion process in stochastic differentiable equation form,

$$dW = b\left(W(t_0)\right)dt_0 + \sigma\left(W(t_0)\right)dz \qquad (12)$$

over any short period of time dt_0, where dz is a Wiener process.

As above, $Y\left(W(t_0)\right)$ is the forward NPV of the project if initiated at time t_0 for a known investment $C(t_0) \geq 0$. Subsequent options, including partial or total reversibility of the stopping decision, are permissible, and these will be priced into $Y\left(W(t_0)\right)$. Initially, to make the problem autonomous we assume that the investment cost is instantaneous and fixed in scale at C.[7] Let $\rho > 0$ be the risk-adjusted discount rate for the opportunity.[8] It is a required rate of return, a rate of time (and uncertainty) preference. At time $t_0 \leq \hat{t}_0$, the project's market (option) value is[9]

$$V\left(W(t_0)\middle|W(\hat{t}_0)\right) = E\left[e^{-\rho(\hat{t}_0 - t_0)}\right]Y\left(W(\hat{t}_0)\right).$$

Using the discount factor notation $E\left[e^{-\rho(\hat{t}_0 - t_0)}\right] \equiv \delta\left(W(t_0)\middle|W(\hat{t}_0)\right)$ of Dixit et al. (1999) we can write

[7] If investment is continuous over a finite interval, C represents the present value of the total investment if all investment must be spent once investment is initiated, or it represents the present value of the minimum discrete lump of investment needed to initiate subsequent investment options.

[8] As is typical in dynamic programming problems we hold the discount rate constant (Insley and Wirjanto 2005).

[9] The value function is changed from Π to V because the domain is now value, rather than time.

$$V\left(W(t_0)\middle|W(\hat{t}_0)\right) = \delta\left(W(t_0)\middle|W(\hat{t}_0)\right)Y\left(W(\hat{t}_0)\right). \tag{13}$$

To reduce notational clutter we hereafter condense $V^*\left(W(t_0)\middle|W(\hat{t}_0)\right)$ to $V(W)$, $V\left(W(\hat{t}_0)\middle|W(\hat{t}_0)\right)$ to $V(\hat{W})$, $V_W\left(W(t_0)\middle|W(\hat{t}_0)\right)$ to $V'(W)$, and $Y(W(t_0))$ to $Y(W)$.

For the broad class of functions *V* and *Y* to which Ito's Lemma can be applied, the expected rate of gain of the option value prior to stopping is

$$\frac{E[dV]}{V(W)dt_0} = \frac{b(W)V'(W) + \frac{1}{2}\sigma^2(W)V''(W)}{V(W)}. \tag{14}$$

The expected rate of gain of forward NPV is

$$\frac{E[dY]}{Y(W)dt_0} = \frac{b(W)Y'(W) + \frac{1}{2}\sigma^2(W)Y''(W)}{Y(W)}. \tag{15}$$

As under certainty, prior to stopping the option value is expected to rise at the rate of discount,

$$\frac{E[dV(W)]}{V(W)dt_0} = \rho. \tag{16}$$

Combining (14) to (16) yields that, prior to stopping,

$$\frac{E[dY]}{dt_0} = \rho Y(W) + b(W)\left(\frac{Y'(W)}{Y(W)} - \frac{V'(W)}{V(W)}\right)Y(W) - \frac{1}{2}\sigma^2(W)\left(\frac{V''(W)}{V(W)} - \frac{Y''(W)}{Y(W)}\right)Y(W). \tag{17}$$

At an interior free boundary the value matching and smooth pasting conditions are

$$V(\hat{W}) = Y(\hat{W}) \qquad (18)$$

and

$$V'(\hat{W}) = Y'(\hat{W}). \qquad (19)$$

From equations (17), (18), and (19), at the stopping point $\hat{W}$,[10]

$$\frac{E[dY]}{Y(\hat{W})dt_0} + \frac{\frac{1}{2}\sigma^2(\hat{W})\left(V''(\hat{W}) - Y''(\hat{W})\right)}{Y(\hat{W})} = \rho\,. \qquad (20)$$

Equation (20) is the second main result of our paper. The right hand side of (20) is the force of interest ρ. The left hand side of (20) is the expected rate of rise of the project's forward NPV as investment is delayed, plus a positive term.[11] Let that positive term be $\alpha(\hat{W}) = \dfrac{\frac{1}{2}\sigma^2(\hat{W})\left(V''(\hat{W}) - Y''(\hat{W})\right)}{Y(\hat{W})}$, so that (20) can be expressed as

$$\frac{E[dY]}{Y(\hat{W})dt_0} + \alpha(\hat{W}) = \rho\,.$$

From our analysis under certainty we would anticipate that $\dfrac{E[dY]}{Y(\hat{W})dt_0} + \alpha(\hat{W})$ is the expected rate of return from waiting to invest, and that at the stopping point the expected return from waiting to invest equals the risk-adjusted force of interest. We will soon show that this is the

[10] Equation (20) can also be derived from the Bellman equation for this type of problem,

$$\tfrac{1}{2}\sigma^2(W)V''(W) + b(W)V'(W) - \rho V(W) = 0$$

by imposing the value matching and smooth pasting conditions, equations (18) and (19), rearranging to put $\rho Y(W)$ on the right-hand side and adding and subtracting $\frac{1}{2}\sigma^2(W)Y''(W)$ on the left-hand sides of the equality.

[11] The term is positive because the second order conditions require that immediately prior to stopping $V^* > Y$ and $|V^{*\prime}| < |Y'|$, which implies that $V^{*\prime\prime} - Y'' > 0$ in the neighborhood of $\hat{W}$ (Takatsuka 2004). Voluntary stopping also requires that $Y > 0$.

case. Given this, equation (20), then, is analogous to the stopping condition under certainty, equation (6), and reduces to equation (6) when $\sigma^2 = 0$.

The behavior of the project around the stopping point also has parallels with the case under certainty. For W in the continuation region but near $\hat{W}$, when

$$\begin{aligned} b(W)\left(\frac{Y'(W)}{Y(W)}-\frac{V'(W)}{V(W)}\right) &> \tfrac{1}{2}\sigma^2(W)\left(\frac{V''(W)}{V(W)}-\frac{Y''(W)}{Y(W)}\right)-\alpha(W) \\ &= \tfrac{1}{2}\sigma^2(W)V''(W)\left(\frac{1}{V(W)}-\frac{1}{Y(W)}\right), \end{aligned} \tag{21}$$

we have, via equation (17),

$$\begin{aligned} \frac{E[dY]}{dt_0 Y(W)}+\alpha(W) &= \rho + b(W)\left(\frac{Y'(W)}{Y(W)}-\frac{V'(W)}{V(W)}\right)-\tfrac{1}{2}\sigma^2(W)\left(\frac{V''(W)}{V(W)}-\frac{Y''(W)}{Y(W)}\right)+\alpha(W) \\ &> \rho. \end{aligned} \tag{22}$$

That is to say, when (21) holds the expected rate of return on delaying the project is greater than the force of interest prior to stopping and falls to the force of interest at stopping, an *r*-percent rule identical to the case under certainty. Our initial explorations indicate that equation (21) appears to hold in many models of economic stopping problems. For example, for call or put options on Brownian motion and geometric Brownian motion processes equation (21) holds when W is drifting towards $\hat{W}$. That there is a widely applicable *r*% rule that is similar to the rule under certainty should provide some comfort to resource and land economists who, by analogy to certainty, commonly quote Hotelling’s, Faustmann’s and Wicksell’s rules as sources of empirical regularities and bases for decisions. Their intuition is approximately correct when they do not and exactly correct when they do include the term α in their calculations of the benefits of waiting to invest.

When, for W in the continuation region but near $\hat{W}$,

$$b(W)\left(\frac{Y'(W)}{Y(W)}-\frac{V'(W)}{V(W)}\right)<\tfrac{1}{2}\sigma^2(W)\left(\frac{V''(W)}{V(W)}-\frac{Y''(W)}{Y(W)}\right)-\alpha(W)$$
$$=\tfrac{1}{2}\sigma^2(W)V''(W)\left(\frac{1}{V(W)}-\frac{1}{Y(W)}\right)<0, \qquad (23)$$

as is the case, for example, when a call option on a geometric Brownian motion has a negative drift, $\frac{E[dY]}{Y(W)dt_0}+\alpha(W)$ is less than the force of interest prior to stopping. It is here that uncertainty requires insights that are not immediately transferable from analyses under certainty.

Returning now to the second-order term α in (20) that differentiates the stopping condition under uncertainty from the condition under certainty, it is useful to consider an alternative stopping condition

$$\frac{E[dY]}{Y(\hat{W}_M)dt_0}=\rho\,. \qquad (24)$$

For an interior solution to this stopping condition, (21) must hold because prior to stopping forward NPV must be expected to rise at a rate greater than ρ. This is therefore an alternative *r*% stopping rule. Equation (24) has been called a *myopic-look-ahead* stopping rule (Clarke and Reed 1989, 1990a), a *stochastic Wicksell rule* (Clarke and Reed 1988), and an *infinitesimal look-ahead* stopping rule (Ross 1970). The rule reflects neoclassical open-loop decision making, comparing investment now with a commitment now to invest next period. Stopping is proposed when the two decisions create the same present value payoff. Open-loop decision making under irreversibility is optimal only when the stochastic process is monotone (Malliaris and Brock 1982, Brock et. al. 1989, Boyarchenko 2004) or more generally when, once a stopping point is reached, the process cannot deviate back into the continuation region (Ross 1970, pp. 188-90). Trees with stochastic prices and stochastic age-dependent growth are an example where open-loop decisions are optimal (Reed and Clarke

1990), as is stopping under certainty. Another example is when *V* is a function of the infimum or supremum of an Ito process (Boyarchenko 2004).

For diffusion processes such as (12) there is a positive probability that the process can drift back into the continuation region once it has reached $\hat{W}$, and the process is not optimally stopped when (24) holds (Brock et al. 1989). Rather, the appropriate *closed-loop* decision is to compare investment now with conditional investment next period. This decision rule is clearly superior to a commitment strategy when the stochastic process can move disadvantageously. This results in additional delay beyond what (24) would suggest; as (20) suggests, the program is immediately stopped when the expected pure postponement flow falls to -α rather than zero. Previous analyses of stopping rules for diffusion processes (including jump processes) have specified only that $\frac{E[dY]}{Y(\hat{W})dt_0} - \rho < 0$ at the stopping point (e.g., Brock et al. 1989; Mordecki 2002), i.e., that expected pure postponement flow is negative. The second-order term -α in (20), derived from the optimality conditions for closed-loop decision making, which include value matching and smooth pasting, is the exact rate to which the pure postponement flow must drop.

It is tempting to identify the adjustment α to the return from waiting as a risk adjustment, where $\frac{1}{2}\left(V''(W) - Y''(W)\right)\big/Y(W)$ is the per unit price of risk and $\sigma^2(W)$ is the quantity of risk. Arnott and Lewis (1979), for example, propose that (24) is the appropriate stopping rule when the investor has rational expectations and is risk-neutral. But equation (20) would hold even in a risk-neutral context, only with *r* replacing ρ. Rather, the adjustment must be some additional value to waiting beyond expected capital gains. In a two-period analysis of intertemporal decision making under irreversibility the adjustment that differentiates the myopic stopping rule from the optimal stopping rule has been identified as quasi-option value (Arrow and Fisher 1974, Fisher and Hanemann 1987, Mensink and Requate 2005). In that analysis quasi-option value is a conceptual lump-sum shadow tax on immediate investment. Equivalently, the correction can be seen as supplying a subsidy to waiting.[12] In our continuous-time case this involves adding the term α to the left-hand-side of (24), creating (20). Because α is a flow we refer to it as *quasi-option flow*. Conrad (1980) has shown quasi-option value to be the expected value of information from delayed decision making, and to the extent that the adjustment α is only effective when $\sigma^2 > 0$, applying Conrad's interpretation to quasi-option flow is appropriate. The left-hand-side of (20) is thus the full rate of return from waiting to invest, consisting of expected capital appreciation plus the value of information. This is compared to the costs of waiting, the force of interest. As Arrow and Fisher (1974, 319) emphasize, examination of irreversible investment under uncertainty "is *not* the overthrow of marginal analysis."

We now explain the functional form of α. In open-loop decision making the global properties of V and Y at the stopping point, as exhibited by V'' and Y'', are irrelevant. Consider, for example, a monotone process where W can only rise and the investment is a call option on $Y(W)$, $Y' > 0$. If (24) holds at $Y(\hat{W})$, then, via the r% rule, waiting will be suboptimal as $\dfrac{E[dY]}{Y(\hat{W}+\Delta h)dt_0} < \rho$ for all possible Δh, where Δh is a function of σ^2. For a non-monotone process the (conditional) decision to invest is influenced by the magnitude of the deviation of $V(\hat{W} - \Delta h)$ from $Y(\hat{W} - \Delta h)$ during a subsequent downward movement in W of magnitude Δh (see Figure 2). Given any Δh, a larger deviation provides a greater incentive to wait for more information prior to investing. To second order, $V'' - Y''$ is proportional to

[12] Fisher and Hanemann's (1987) equation (9) is $V*(0) - \left[V*(1) - V_q\right] = \hat{V}(0) - \hat{V}(1)$, where the quasi-option value V_q is a tax applied to the payoff from immediate development, $V*(1)$. The optimal decision rule could equivalently be induced by providing a subsidy of V_q to the payoff from waiting, $V*(0)$.

$V(\hat{W}-\Delta h)$ - $Y(\hat{W}-\Delta h)$, and the global properties of the curvature of *V* and *Y* around $\hat{W}$ impact the now closed-loop stopping condition.

The discussion thus far supports the following proposition, which holds for all values of $\sigma^2 \geq 0$:

> ***Proposition 1:*** *At the optimal stopping point in an interior solution the rate of return from waiting to invest is equal to the force of interest. The rate of return from waiting to invest includes expected capital gains of forward project NPV and any information value that we call quasi-option flow.*

When equation (21) holds an additional proposition obtains:

> ***Proposition 2:*** *Prior to the stopping point, rate of return from waiting to invest is expected to exceed the force of interest.*

Appendix 2 applies these propositions to three canonical stopping problems. To facilitate the graphical presentations in each case the stopping condition is modified to compare the expected rate of rise of forward NPV against an adjusted force of interest $\rho^* = \rho - \alpha$. Figure 3 depicts the rule, modified in this fashion, for the case of geometric Brownian motion studied in Appendix 2.B. There is a strong similarity between the stopping rule depicted in Figure 3, in the value domain, and the same rule depicted under certainty in Figure 1, in the time domain. Fisher and Hanemann (1987) and Kennedy (1987) claim that, because quasi-option value cannot be estimated separately as an input to the analysis, it is only of conceptual value and not useful as a tool for optimal stopping. Appendix 2 shows that quasi-option *flow* is in fact estimable as a separate adjustment to the flow opportunity cost of stopping when the option value *V* has a closed-form general solution. In the geometric Brownian motion example (Appendix 2.B and Figure 3), quasi-option *flow* adjusts the 14% rate of discount ρ downward by four percentage points.

Our analysis thus far has been for a simple single-factor stochastic process. More general single factor stochastic processes allow time to enter as a state variable, either via a finite option termination date *T* or through the stopping cost $C(t_0)$ or a time-dependent discount rate $\rho(t_0)$, or via the addition of time to the drift or variance terms in the continuous portion of the stochastic process for the underlying variable,

$$dW = b(W(t_0),t_0)dt_0 + \sigma(W(t_0),t_0)dz\,. \tag{25}$$

There may also be an exogenous flow $\Theta(t_0)$ associated with delay, such as when there are explicit payments made ($\Theta(t_0)$ <0) or receipts accrued ($\Theta(t_0)$ >0) while keeping the option alive. Using the same derivation as previously, the stopping condition for an interior stopping point is

$$\frac{E[dY]}{Y(\hat{W},\hat{t}_0)dt_0} + \frac{\Theta(\hat{t}_0)}{Y(\hat{W},\hat{t}_0)} + \frac{\left(V_{t_0}(\hat{W},\hat{t}_0;T) - Y_{t_0}(\hat{W},\hat{t}_0)\right)}{Y(\hat{W},\hat{t}_0)} + \frac{\frac{1}{2}\sigma^2(\hat{W},\hat{t}_0)\left(V_{WW}(\hat{W},\hat{t}_0;T) - Y_{WW}(\hat{W},\hat{t}_0)\right)}{Y(\hat{W},\hat{t}_0)} = \rho(\hat{t}_0). \tag{26}$$

The partial derivatives V_{t_0} and Y_{t_0} are typically taken to be depreciation (V_{t_0}, Y_{t_0} < 0) or appreciation (V_{t_0}, Y_{t_0} > 0) due to worsening or improving project economics over time (Dixit and Pindyck 1994, 205-207). In the case of a finite-lived option V_{t_0} also includes the decay in the value of the option as the time to expiry draws nearer. With time now a state variable, and for other processes such as mean reversion, there is no closed-form solution for the option. As a result, quasi-option flow cannot be estimated and utilized to obtain the optimal stopping point. Nevertheless, equation (26) shows that at any interior stopping point, even for finite-lived options, the rate of return from delaying action, including the value of information, equals the intertemporal opportunity cost of waiting, the force of interest.

4. Discussion: Theoretical Issues

Under certainty the *r*% rule has the investor wait until the rate of rise of the forward NPV falls to a force of interest before investing. We have shown that a similar rule can hold under uncertainty. That rule has the investor waiting until the total rate of return from delay, which includes the expected rate of rise of the forward NPV plus the value of information from waiting, falls to the force of interest. The rule applies to investment problems that can include any number of market conditions and subsequent investment options.

There are theoretical and practical benefits to seeing stopping under uncertainty as a condition involving opportunity costs and benefits akin to the problem under certainty. We first use stopping condition (20) and, where applicable, the associated *r*% rule to reexamine certain common intuitive notions about stopping under uncertainty presented in the academic and practitioner literature.

The first notion is that waiting to invest is only valuable under uncertainty, or at least "driven by uncertainty," due to information gained through delay.[13] Let the program be stopped at myopic point $\hat{W}_M$ and the option value under the myopic stopping rule (24) be

$$V_M\left(W\right) = E\left[e^{-\rho(t_0^M - t)}\right]Y\left(\hat{W}_M\right) = \delta\left(W\middle|\hat{W}_M\right)Y\left(\hat{W}_M\right).$$

[13] Fisher (2000, 203), for example, states that "…the option to postpone the investment has value only because the decision-maker is assumed to learn about future returns by waiting." Similar statements — "The value of waiting is driven by uncertainty" (Amram and Kulatilaka 1999, 179); "The deferral option, or option of waiting to invest, derives its value from reducing uncertainty by delaying an investment decision until more information has arrived" (Brach 2003, p. 68); "…the simple NPV rule…is rarely optimal, since delaying can yield valuable information about prices and costs" (Moyen et al. 1996, 66) — abound.

Let stopping rule (24) have an interior solution $t_0^M > t$. This means that prior to t_0^M point expected pure postponement flow is positive.[14] With this, (21) holds and the parallels with stopping under certainty are assured. Generalizing(Mensink and Requate's (2005) two-period example, the premium or value of waiting under this stopping rule is pure postponement value

$$P(W) \equiv V_M(W) - Y(W). \tag{27}$$

Quasi-option value as the difference between an interior solution to the Bellman equation using stopping rule (20) and the solution using the myopic stopping rule (24),

$$Q(W) \equiv V(W) - V_M(W) \geq 0. \tag{28}$$

The option premium (the value of waiting) is

$$O(W) = V(W) - Y(W) = Q(W) + P(W). \tag{29}$$

Equation (29) reveals that the value of waiting is in this case the sum of pure postponement and quasi-option values. Only in the domain where $Q > P$ can uncertainty be said even to dominate the value of waiting. At the intertemporal margin, when expected pure postponement flow is positive, delay is warranted regardless of whether or not such delay provides information value. Uncertainty, or more specifically the value of information, drives delay at the margin only when expected pure postponement flow is negative. That is to say, uncertainty is the sole reason for delay only when either equation (21) fails to hold or $\rho > \frac{E[dY]}{Y(W)dt_0} > \rho - \alpha(W)$.

Mensink (2004) and Mensink and Requate (2005) have pointed out specific confusions between quasi-option value and pure postponement value in two-period models of environmental preservation. Our analysis reveals that the terms are useful in describing the

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[14] If $\frac{E[dY]}{Y(W)dt_0} - \rho \leq 0$ it will be optimal to stop immediately under the myopic rule.

motivations for delay in stopping problems more broadly. We illustrate this point in a numerical call option example. Given a geometric Brownian motion process and the resultant option value given in appendix equation (A7) we can write the option value as

$$\left(\frac{W}{\hat{W}}\right)^{\beta}\left(\hat{W}-C\right)=\left(W-C\right)+\left(\left(\frac{W}{\hat{W}_M}\right)^{\beta}\left(\hat{W}_M-C\right)-\left(W-C\right)\right)+\left(\left(\frac{W}{\hat{W}}\right)^{\beta}\left(\hat{W}-C\right)-\left(\frac{W}{\hat{W}_M}\right)^{\beta}\left(\hat{W}_M-C\right)\right). \qquad (30)$$

The first term on the right-hand side is NPV, the second term is pure postponement value (which is positive if $W < \hat{W}_M$, negative if $W > \hat{W}_M$, or zero if $W = \hat{W}_M$), and the third term is quasi-option value (which is non-negative given $\hat{W}_M \le \hat{W}$). Figure 4 shows the pure postponement value and quasi-option value of waiting. The dashed line gives $\left(\frac{W}{\hat{W}_M}\right)^{\beta}\left(\hat{W}_M-C\right)$, the value of the program if it is stopped using equation (24). At the open-loop stopping point, $\hat{W}_M$ = 1.56, and expected pure postponement flow falls to zero (and hence, under an open-loop strategy, the program is stopped). For W < 1.56 expected pure postponement flow is positive and pure postponement value is positive, with forward NPV rising at greater than ρ. For W > 1.56 expected pure postponement flow is negative and pure postponement value is negative: the asset is rising at less than ρ. Only when W > 1.42 is Q > P and quasi-option value the main proportion of the option premium gained from investment delay, and only when W > 1.56 can one say that waiting is solely due to uncertainty.

When equation (20) is used as the stopping rule, $\hat{W}$ = 2.00, at which point value matching implies that the option premium is zero, $O(\hat{W}) = 0$, and quasi-option value exactly offsets the negative pure postponement value, $Q(\hat{W}) = -P(\hat{W})$. Smooth pasting implies that $O'(\hat{W}) = 0$, so that $Q'(\hat{W}) = -P'(\hat{W})$ at $\hat{W}$.

The r% rule also makes clear that Y and σ^2 are not always sufficient statistics by which to assess the optimal time to act. Luehrman's (1998) practical guide to investment timing under uncertainty suggests investors plot their investment opportunities in NPV and volatility space. Investors are encouraged to invest immediately if project NPV is positive and project volatility is low, whereas they should wait if volatility is high. Indeed, we have seen major corporations use this heuristic, which ignores the *r*% rule, when timing investment options. The approximation is possibly appropriate when equation (23) holds and the economics of the project are likely to worsen in the immediate future. In that case interior solutions are unlikely at low volatilities. When (21) holds, on the other hand, considerations of pure postponement flow should also be taken into account, regardless of the level of Y and σ^2. The point is perhaps obvious, but to our knowledge has never been formalized via a specific stopping rule.

Another intuition is that uncertainty always creates a stricter investment hurdle $\hat{W}$. Stopping condition (20) shows that this is not so. Increasing uncertainty will indeed impact α. But it will also impact expected pure postponement flow, $\frac{E[dY]}{Y(W)dt_0} - \rho$, by increasing ρ and increasing or decreasing $Y(W)$ depending on whether $Y(W)$ contains subsequent options. For the call option example depicted in Figure 3, under certainty the discount rate on the asset falls to the riskless rate, 6%, and $\hat{W} = 6$. For $2 \le W < 6$ the program is continued under certainty, whereas under uncertainty it is stopped immediately since $\hat{W} = 2$. Sarkar (2003) shows for a specific case that uncertainty may increase or decrease the investment hurdle, in part depending on the sensitivity of $Y(W)$ to uncertainty. Equation (20) shows that the indeterminacy is general. The faulty intuition again arises from limiting attention to value of information term α as a reason for delay.

Stopping condition (20) also provides insights into the role of irreversibility in delaying stopping. With irreversibility the cause of delay beyond the myopic stopping rule (Dixit and Pindyck 1994, 6), diminished irreversibility will be reflected as a decrease in $V^{*\prime\prime} - Y^{\prime\prime}$.

Appendix 3 shows that in the limit, as investment becomes completely reversible, $V^{*\prime\prime} - Y^{\prime\prime} = 0$. The Appendix also shows that a repeated option to invest, on the other hand, does not reduce the reversibility of the initial investment, even though it accelerates that investment. The acceleration is simply due to an adjustment to pure postponement flow. The term α in (20) may also be thought of, then, as an "irreversibility factor" proportional to the irreversibility of the investment action in the face of subsequent options to undo that action.

Such considerations of reversibility in optimal stopping have implications for intuitive perceptions of equilibrium. For example, a ranking of heterogeneous discount rates, ρ_i, where i indexes the project, has been proposed as being sufficient to order the timing of mineral production (Malliaris and Stefani 1994), in line with the economic intuition under certainty that higher discount rates are a result of higher opportunity costs of waiting (Stiglitz 1976). But the r% rule shows that heterogeneity in the information from delay also comes into play in timing entry. Investments that are less irreversible will have lower value-of-information benefits to delay, and through this a smaller adjustment α in (20). The determination of investment timing is thus the outcome of a complex sectorial equilibrium involving price paths, interest rates, and the value of information, with expected rates of capital gain from waiting being compared against ($\rho_i - \alpha_i$) rather than just ρ_i, though the stopping condition (20) that is endogenous to that equilibrium holds for both certainty and uncertainty, perfect competition and imperfect competition, and indeed in all cases for all assets.

Finally, since $\dfrac{E[dY]}{Y(\hat{W})dt_0} < \rho$, stopping condition (20) supports the notion that projects must at some point exhibit a rate-of-return shortfall, or notional dividend yield, for new investment to be forthcoming under uncertainty (Davis and Cairns 1999). Prices in backwardation are one mechanism by which the market can bring forth investment (Litzenberger and Rabinowitz 1995).

5. Discussion: Issues in Practice

The *r*% rule and the intuition that it reveals may also influence practice. Practitioners have been slow to adopt explicit optimal stopping algorithms when timing investment decisions under uncertainty (Triantis 2005). Berk (1999) suggests that this is because the opportunity to delay depends idiosyncratically on the nature of the uncertainty and the decision, preventing the derivation of a simple rule that could replace the traditional NPV rule. Timing rules are instead presented in the literature as an upper or lower boundary on project value or price (e.g., Dixit and Pindyck 1994) or as a present value index (Moore 2000). Copeland and Antikarov (2005, 33) note, however, that relying on such presentations is unsatisfactory:

> The academic literature about real options contains what, from a practitioner's point of view, is some of the most outrageously obscure mathematics anywhere in finance. Who knows whether the conclusions are right or wrong? How does one explain them to the top management of a company?

One does not use what one does not understand, and there has been little success in explaining the intuition of optimal stopping under uncertainty. Practitioners themselves suggest that optimal stopping rules will only be adopted if the can be seen as a complement to, rather than a replacement of, traditional NPV analysis (e.g., Woolley and Cannizzo 2005). Berk (1999), to this end, derives a NPV-based optimal stopping rule for the case of stochastic interest rates and cash flows that are riskless or where there is no resolution of uncertain cash flows by waiting. Boyarchenko (2004) derives an adjusted NPV decision rule when waiting updates information about future cash flows that are geometric Lévy processes. The representation of stopping under uncertainty as an *r*% rule supports these advances by providing a link to the intuition many practitioners already have from related timing rules under certainty. For example, the intuitive attractiveness of pure postponement flow is already leading to comparisons of expected changes in NPV with the opportunity cost of capital when deciding when to develop a mine or harvest a stand of trees, as in myopic rule (24) (e.g., Torries 1998, 44, 75; Yin 2001, 480). Land owners also appear to recognize and time development according to pure postponement flow (Arnott and Lewis 1979; Holland, Ott, and Riddiough 2000). Stopping condition (20) and the *r*% rule show that the value of information from waiting must also be taken into account, and that comparisons of expected changes in NPV with the opportunity cost of capital will not yield sufficient patience in cases where the investment is irreversible. The adjustment, however, is now an adaptation of a rule that is already in effect, rather than presented as a completely new way of viewing stopping.

The *r*% rule is also helpful in understanding why waiting should at some point stop, even though further waiting would reveal even more information about future levels of the state variable. Scrapping options are particularly awkward in this regard, since decision makers are expected to incur operating losses prior to stopping. The intuition given is usually that economic conditions could get better, though due to the option to scrap they can't get much worse (e.g., McDonald and Siegel 1986). But this adage is true even beyond the optimal stopping point. Why stop when operating losses reach a particular level? Recursive explanations, such as with a lattice, are of no use since the options to abandon is often perpetual. We illustrate the intuition provided by the *r*% rule using Clarke and Reed's (1990b) model of the optimal time to irreversibly abandon a producing oil well with fixed operating costs but declining production.

In Clarke and Reed's study, both oil price $P(t)$ and extraction rate $Q(t)$ evolve as geometric Brownian motions:

$$d\pi = b dt + \sigma_\pi dw_\pi , \qquad (31)$$

$$dq = -\delta dt + \sigma_q dw_q , \;\; \delta > 0, \qquad (32)$$

where $\pi(t) = \log P(t)$ and $q(t) = \log Q(t)$. Let $z(t) \equiv \pi(t) + q(t)$ be the logarithm of revenue. Then, by Ito's lemma, $z(t)$ is a Brownian motion with drift $-d = \delta - b$ and variance $\sigma^2 = \sigma_\pi^2 + \sigma_q^2 + 2\sigma_{\pi q}$ where $\sigma_{\pi q}$ is the covariance between the log price and log extraction processes. At time T the forward NPV of abandonment is

$$Y\left(z(T)\right) \equiv A - Be^{z(T)} \qquad (33)$$

where A is the present value of operating costs saved net of the abandonment cost and $Be^{z(T)}$ is the after-tax expected present value of revenues foregone. A and B are autonomous functions. The goal is to determine the revenue level $e^{\bar{z}}$ that induces optimal abandonment timing.

Clarke and Reed show that the value of the option to abandon for $z(t) > \bar{z}(t)$ is of the form

$$V(z) = Y(\bar{z}) \exp\{\Theta(z - \bar{z})\}, \qquad (34)$$

where $\Theta \equiv [d - \sqrt{d^2 + 2\rho\sigma^2}\,]/\sigma^2 < 0$. Invoking stopping condition (20) equals

$$\begin{aligned}
\frac{E[dY]}{Y(z)dT} + \alpha(z) &= \frac{-dY'(z) + \tfrac{1}{2}\sigma^2 Y''(z)}{Y(z)} + \frac{\tfrac{1}{2}\sigma^2\left(\Theta^2 Y(z) + Be^z\right)}{Y(z)} \\
&= \frac{dBe^z - \tfrac{1}{2}\sigma^2 Be^z}{Y(z)} + \frac{\tfrac{1}{2}\sigma^2\left(\Theta^2 Y(z) + Be^z\right)}{Y(z)} \\
&= \frac{dBe^z}{Y(z)} + \tfrac{1}{2}\sigma^2\Theta^2 \\
&= \rho.
\end{aligned}$$

Solving,

$$e^{\bar{z}} = \frac{A(\rho - \frac{1}{2}\sigma^2\Theta^2)}{B(d + \rho - \frac{1}{2}\sigma^2\Theta^2)} = \frac{A\Theta}{B(\Theta - 1)}\,. \tag{35}$$

This is equation (18) in Clarke and Reed, which they derive from the usual value-matching and smooth-pasting conditions.

We now demonstrate the intuition afforded by the *r*% stopping rule. Using a gross proceeds tax rate of 16.5%, an abandonment cost of 0, a fixed per-period operating cost of 33.53, a rate of drift of revenues of -10% (d = 0.10), a revenue volatility σ^2 of 0.03, and a discount rate of 4% (all taken from Table 3 of Clarke and Reed) the value of the put option is expected to improve over time as revenues fall towards the stopping point. Given these parameters (21) holds, and the *r*% rule is appropriate. The traditional NPV stopping rule that would have the project abandoned while after-tax net income is still positive, at 71.25, and while the forward NPV of abandonment expected to rise at 11.2%. This is clearly not optimal given the 4% discount rate. From equation (35) abandonment should optimally occur when $e^{\bar{z}}$ = 34.46, at which point after-tax net income flow is 0.835 x 34.46 - 33.53 = -4.76. The forward NPV of abandonment at the shutdown point is

$Y(\bar{z}) = A - Be^{\bar{z}} = 838.25 - 6.68 \text{ x } 34.46 = 608.06$, and $\frac{E[dY]}{Y(\bar{z})dT} = \rho^* = \rho - \alpha = 3.2173\%$.

For revenue levels of greater than 34.46 the forward NPV of abandoning is less than 608.06 and expected to rise at a rate greater than 3.2173%, as shown in Figure 5.

The myopic stopping rule (24) gives

$$e^{\bar{z}_M} = \frac{A\rho}{B(d + \rho - \frac{1}{2}\sigma^2)} = 40.16\,, \tag{37}$$

at which point the forward NPV is 570.01. The revenue level of 40.16 is also the point where after-tax net income is zero. For revenue levels of greater than 40.16 the forward NPV from abandonment is expected to rise at greater than 4%, and waiting generates positive expected pure postponement flow. As revenues fall below 40.16 after-tax net income becomes negative and pure postponement flow becomes negative as the forward NPV from abandonment is ecpected to rise at less than 4%. This myopic stopping point is a natural stopping point that practitioners would find more intuitive than the Marshallian stopping point. But continued waiting has value due to quasi-option flow. Only when the expected rate of rise of the forward NPV of abandoning falls to 3.2173%, at a revenue level of 34.46, is the opportunity cost of waiting—negative pure postponement flow—enough to offset the quasi-option flow opportunity benefits of waiting.

Practitioners will easily appreciate, through the concept of pure postponement value, why it is not optimal to abandon as soon as it is profitable to do so; if the NPV of abandonment is rising at greater than the discount rate is it optimal to wait. Yet they may still be uncomfortable continuing to operate while incurring negative net operating income, which is what the *r*% rule suggests. The optimal stopping point becomes more tenable if they see that the NPV of abandonment is still expected to rise at only slightly less than the discount rate during the final moments of the waiting period, and that the full benefits of waiting include a term $\alpha(z) = \dfrac{\frac{1}{2}\sigma^2\left(\Theta^2 Y(z) + Be^z\right)}{Y(z)}$ representing the information value gained by delay.

6. Conclusions

Timing rules based on comparisons of mutually exclusive projects starting at different times lead to optimal timing decisions under certainty, just as they do under uncertainty. The substantive difference under uncertainty is one of method; problems are usually solved in the value domain rather than the time domain. By staying within the time domain we have shown that there is a consistent *r*% stopping rule that applies to a wide class of problems under both certainty and uncertainty. In that rule investment takes place only once the rate of return from waiting to invest falls to the force of interest. Under certainty the opportunity cost of stopping is lost capital gains on the project. Under uncertainty the opportunity cost of stopping is increased by the flow value of information gained by delaying an irreversible action. The degree of adjustment is proportional to the irreversibility of the investment.

Seeing the stopping condition under uncertainty as having close economic parallels to the case under certainty reveals that the theory of investment under uncertainty is an incremental generalization of, not a qualitative break from, the traditional theory of investment under certainty. For what we estimate to be a wide class of economic problems, the stopping rule under certainty is simply the limiting case of uncertainty as volatility goes to zero.

Appendix 1. Serving Wine, Certainty

Suppose that a connoisseur has a bottle of wine that can provide one util if served immediately or can be stored costlessly and served at time $t_0 > 0$ to yield $\exp(\sqrt{t_0})$ utils. Let the instantaneous utility-discount rate be $r = 0.20$ and let $t = 0$. If the wine is consumed at t_0, the current value of the wine is

$$\Pi(0,t_0) = D(0,t_0)W(t_0) = \exp(-rt_0)\exp(\sqrt{t_0})\,. \qquad (1)'$$

Equation (2) implies that the connoisseur waits until $\hat{t}_0 = 1/(4r^2)$ to serve the wine, even though serving it now would provide a positive benefit of one util. We confirm equation (6) by observing that

$$\frac{W'(\hat{t}_0)}{W(\hat{t}_0)} = \frac{0.5(\hat{t}_0)^{-0.5}\exp(\sqrt{\hat{t}_0})}{\exp(\sqrt{\hat{t}_0})} = 0.5(2r) = r\,.$$

If $r = 0.20$, for example, the wine is served at $\hat{t}_0 = 1/(4r^2) = 6.25$. At that time it is worth $W(\hat{t}_0) = \exp(\sqrt{\hat{t}_0}) = 12.1825$ utils and has a current value of $\exp(-r\hat{t}_0)\exp(\sqrt{\hat{t}_0}) = 3.49$ utils. The option premium or pure postponement value associated with the option to store the wine is 3.49 – 1 = 2.49 utils, a full 71% of the wine's current value.

Appendix 2. Optimal Stopping for Particular Stochastic Processes

A. Arithmetic Brownian Motion. Let $W(t_0)$ follow the Brownian motion process

$$dW = b\,dt_0 + \sigma dz\,, \qquad (12)'$$

$b > 0$.[15] Such a process for birth value can be due to stochastic prices, operating costs, closure costs, or any combination of these.

[15] If $b = 0$ the process for NPV is stationary in this example, causing the r% rule to be ineffective. If $b < 0$ the forward NPV is expected to fall. That rate of decline rises to the adjusted force of interest at the stopping point.

Typically, (12)′ represents the diffusion process for forward value only on an open set called the continuation region. For a call option the upper bound to this continuation region is free, while at a lower bound $\underline{L}$ there is some condition such as $V^*(\underline{L}) = 0$ (Brock et al. 1989). To keep matters tractable we assume that $\underline{L} = -\infty$. Without loss of generality let development be costless ($C = 0$) and $Y = W$. Also let ρ be the constant risk-adjusted discount rate used to discount the investment payoff $\hat{W}$.

The market value of a perpetual opportunity to invest satisfies the linear second-order linear differential equation,

$$\rho V(W) - bV'(W) - \tfrac{1}{2}\sigma^2 V''(W) = 0. \tag{A1}$$

The solution to (A1) is of the form

$$V(W) = A_1 e^{\lambda W} + A_2 e^{\mu W} \tag{A2}$$

where $\lambda > 0$ and $\mu < 0$ are roots of the characteristic equation

$$\tfrac{1}{2}\sigma^2\gamma^2 + b\gamma - \rho = 0. \tag{A3}$$

In the absence of any holding costs and coerced stopping at a finite lower bound $\underline{L}$, $\lim_{W\to-\infty} V(W) = 0$. From this, $A_2 = 0$. If the program is voluntarily stopped at $\hat{W}$, $V(\hat{W}) = \hat{W}$ so that $A_1 = \hat{W}e^{-\lambda\hat{W}}$ and

$$V(W) = \delta(W \,|\, \hat{W})\hat{W} = e^{-\lambda(\hat{W}-W)}\hat{W}. \tag{A4}$$

As of this point the solution for $\hat{W}$ is not evident. The smooth pasting condition

$$V'(\hat{W}) = \lambda\hat{W}e^{-\lambda(\hat{W}-\hat{W})} = \lambda\hat{W} = 1, \tag{19$'$}$$

produces $\hat{W} = \lambda^{-1} = \left(\dfrac{-b + \sqrt{b^2 + 2\sigma^2\rho}}{\sigma^2}\right)^{-1}$.

We now perform the stopping calculation using equation (20). The first term on the left-hand side of (20) equals $\frac{b}{W}$, the expected rate of rise of forward NPV. Given equation (A4) the second term on the left-hand side of (20) is $\alpha = \frac{1}{2}\lambda^2\sigma^2$. Under the *r*% rule the program is stopped when

$$\frac{E[dY]}{Y(W)dt_0} = \frac{b}{W} = \rho - \alpha = \rho - \tfrac{1}{2}\lambda^2\sigma^2 = \lambda b\,, \qquad (20)'$$

the second equality following from (A3). Solving (20)′, $\hat{W} = \lambda^{-1} > 0$. Note that the solution has not explicitly invoked the smooth pasting condition; it is embedded in the *r*% rule.

From (A4) and (20)′ the value of the option is:

$$e^{-\lambda(\hat{W}-W)}\hat{W}, \qquad W \le 0 \text{ or } \frac{b}{W} > \rho - \tfrac{1}{2}\lambda^2\sigma^2;$$

$$W, \qquad W > 0 \text{ and } \frac{b}{W} \le \rho - \tfrac{1}{2}\lambda^2\sigma^2.$$

As was the case under certainty, $V(W) - W > 0$ is the option premium when $W < \hat{W}$. Under uncertainty it is a combination of both pure postponement value and quasi-option value. For $W > 0$, expected pure postponement flow is positive when $\frac{b}{W} > \rho$, or when $W < \hat{W}_M = \frac{b}{\rho}$. Delay at the intertemporal margin is exclusively due to quasi-option value only when $\frac{b}{\rho - \frac{1}{2}\lambda^2\sigma^2} > W > \frac{b}{\rho}$.

B. Geometric Brownian Motion. Consider a state variable whose value follows a geometric Brownian motion with constant rate of drift *b*, represented in differential form over a small time step dt_0 as

$$dW = bWdt_0 + \sigma W dz\,, \qquad (20)''$$

$b > 0$. Let the required rate of return on W be represented by u and be constant and greater than b. Also let the risk-free rate be represented by r. With b constant the investment cost C must be positive to avoid bang-bang now/never stopping solutions. Let the forward NPV be

$$Y = W - C . \tag{A5}$$

For call-option type investments the option value is of the form (Dixit et al. 1999)

$$V(W) = \delta(W \big| \hat{W})\hat{Y} = \left(\frac{W}{\hat{W}}\right)^{\beta} \hat{Y} . \tag{A6}$$

Dixit et al. (1999) use value matching and smooth pasting conditions to solve $\hat{W} = \dfrac{\beta}{\beta - 1} C$,

where $\beta > 1$ is the positive root of the characteristic equation

$$\tfrac{1}{2}\sigma^2 \gamma(\gamma - 1) + b\gamma - \rho = 0 \tag{A7}$$

and $\rho = r + \beta(u - r) > u$ (McDonald and Siegel 1986).

Using the r% stopping rule, from (A5) and (A7) the first term on the left-hand side of (20) equals $\dfrac{bW}{W - C}$. From (A6) the second term on the left-hand side of (20) is $\alpha = \tfrac{1}{2}\sigma^2 \beta(\beta - 1)$, where $\sigma^2(W) = \sigma^2 W^2$. At $\hat{W}$ the r% rule gives

$$\frac{E[dY]}{Y(\hat{W})dt_0} = \frac{b\hat{W}}{\hat{W} - C} = \rho - \alpha = \rho - \tfrac{1}{2}\sigma^2 \beta(\beta - 1) . \tag{20$''$}$$

From (A7),

$$\rho - \tfrac{1}{2}\sigma^2 \beta(\beta - 1) = b\beta ,$$

yielding $\hat{W} = \dfrac{\beta}{\beta - 1} C$. Using the r% rule the option value is

$$\begin{array}{ll} \left(\dfrac{W}{\hat{W}}\right)^{\beta} \hat{Y} , & W \le C \text{ or } \dfrac{bW}{W - C} > \rho - \tfrac{1}{2}\sigma^2 \beta(\beta - 1) \\[2ex] Y , & W > C \text{ and } \dfrac{bW}{W - C} \le \rho - \tfrac{1}{2}\sigma^2 \beta(\beta - 1) . \end{array}$$

Figure 3 depicts this stopping problem for specific parameter values. In this case $\hat{W} = 2$ and $\hat{Y} = \hat{W} - C = 1$. The pure postponement flows and pure postponement values associated with this stopping problem are discussed in the text.

C. Combined Process. Extending Appendix example B, if W follows a combined geometric Brownian motion with an independent downward Poisson jump of known percentage ψ and arrival rate η (Dixit and Pindyck 1994, pp. 167-173), the r% stopping rule can be shown to be

$$\frac{E[dY]}{Y(W)dt_0} = \frac{(b-\eta\psi)W}{W-C} = \rho - \alpha = \rho - \tfrac{1}{2}\sigma^2\beta(\beta-1), \qquad (20)'''$$

where β is now the positive solution iteratively satisfying

$$\tfrac{1}{2}\sigma^2\gamma(\gamma-1) + b\gamma - (\rho+\eta) + \eta(1-\psi)^\gamma = 0 \qquad (A8)$$

and $V(0) = 0$. In the absence of a jump process ($\eta = 0$) equations (20)′′′ and (A8) revert to (20)′′ and (A7). When $\eta > 0$ the jump process is seen to lower the expected rate of increase of forward NPV in equation (20)′′′ and thereby lower expected pure postponement flow. It also adjusts the value of β in the flow shadow tax term. For example, for $\psi = 1$, $\eta > 0$ causes β to increase, and so causes the shadow tax adjustment to rise. The reduced pure postponement flow reduces the opportunity cost of stopping, while the increased quasi-option flow reduces the opportunity benefit of stopping by imposing a higher shadow tax on the rate of discount. In the examples given in Dixit and Pindyck (Table 5.1) the reduction in pure postponement flow wins out, and stopping is advanced as a result of the jump process.

Appendix 3. Optimal Stopping and Reversibility

The second-order term α in stopping condition (20) reflects the value of information attributed to a conditional investment rule as compared with an unconditional commitment to invest. Information has value because of the irreversibility of the investment decision. Under partial reversibility, where there are subsequent options for disinvestment and then reinvestment, the value of information from waiting is diminished. This appendix demonstrates that in the limit, as the investment becomes completely reversible, $\alpha = 0$ and the myopic stopping rule is optimal. It also shows that repeated options to invest are no less irreversible than a single option to invest, even though they advance investment timing. To conserve space we borrow from previous models in making these demonstrations.

Consider the investment opportunity in Section II of Brennan and Schwartz (1985). There, the firm has an infinite set of options to invest in (open) and disinvest in (close) a project that has already been developed. The problem is autonomous. Set operating costs equal to zero to focus only on investment timing options. Also set all taxes equal to zero to simplify the analysis. For fixed investment cost k_2 a firm can invest in a revenue stream with present value qs/κ, where q is fixed per period output, s is the unit price, which follows a geometric Brownian motion, and κ is the rate of convenience on the unit price. In the Brennan and Schwartz model the firm can undo its investment decision by paying k_1, reinvest by paying k_2, and so on. A slight modification makes this a model of partially reversible investment. Define reversibility as the ability to sell each incidence of capital investment k_2 and *receive* some portion of it k_1 upon closure, which can be introduced into the Brennan and Schwartz model by setting $k_1 = -\phi k_2$, $0 \le \phi \le 1$. Now, as opposed to disinvestment as being an expense, it is a partial or total refund of investment cost k_2. As in Dixit and Pindyck (2000), complete irreversibility is defined as $\phi = 0$, and complete reversibility as $\phi = 1$.

In Brennan and Schwartz's notation, the functional form for the value of the option to invest is

$$w(s) = \beta_1 s^{\gamma_1}, \quad \beta_1 > 0,\ \gamma_1 > 0, \tag{A9}$$

and the functional form for the forward NPV is[16]

$$v(s) = \beta_4 s^{\gamma_2} + \frac{qs}{\kappa} - k_2, \quad \beta_4 > 0, \ \gamma_2 < 0. \tag{A10}$$

The term $\beta_4 s^{\gamma_2} \geq 0$ is the value of any reversibility of the investment decision (the value of the put option that one obtains by investing).

Define s_2^* as the optimal stopping point for investment and $s_1^* \leq s_2^*$ as the optimal stopping point for disinvestment. There is no closed-form solution for s_2^* and s_1^* when the two are not equal. Brennan and Schwartz provide the algorithm for a numerical solution. Using the solution values for s_2^* Figure A3.1 provides the calculated values of $\rho^* = \rho - \alpha$ for various levels of reversibility ϕ given $\rho = 0.10$ and other parameter values taken from Brennan and Schwartz. The parameter values satisfy equation (21), and the r% rule holds; the rate of rise of forward NPV falls to ρ^* at the investment stopping point. Under complete reversibility $\phi = 1$ and $s_1^* = s_2^*$, from which it can be shown algebraically that $V^{*\prime\prime}(s_2^*) - Y''(s_2^*) = 0$; the myopic stopping rule is optimal as there is no value of information from waiting beyond the strike point indicated by that rule. This is also evident in the calculated values in the Figure, where $\rho^* = \rho$ when $\phi = 1$.

We now show that repeated options do not necessarily reduce the irreversibility of investment and thereby reduce the value of information from waiting. That is, repeated options are not necessarily the same as reversibility. To illustrate this we use Malchow-Møller and Thorsen’s (2005) single-factor model of the option to replace stationary existing productivity level Θ with the productivity level of a stochastic exogenous technology, θ. That technology evolves as a geometric Brownian motion with rate of drift $0 < \alpha < \rho$. At the

[16] There is a typographical error in the paper. The correct form for β_4 is $\dfrac{ds_1^*(\gamma_1 - 1) + e\gamma_1}{(\gamma_2 - \gamma_1)s_1^{*\gamma_2}}$.

optimal time of replacement $\hat{\theta} = \lambda\Theta$, where $\lambda \geq 1$. Replacement costs $c\theta$, $c > 0$. Using Malchow-Møller and Thorsen's notation the value of the option to invest is

$$V(\theta,\Theta) = \Theta A_1 \left(\tfrac{\theta}{\Theta}\right)^{a_1} \qquad \text{A(11)}$$

where $a_1 > 1$ is the usual positive root of the quadratic equation and $A_1 = -\frac{\lambda^{-a_1}}{\rho(1-a_1)} > 0$.

In the single option case $Y = \theta/\rho - c\theta - \Theta/\rho$. The r% rule is to invest when[17]

$$\Theta + \alpha\theta(\tfrac{1}{\rho} - c) + \tfrac{1}{2}\sigma^2 \tfrac{a_1}{\rho}\Theta = \rho\theta(\tfrac{1}{\rho} - c). \qquad (24)'$$

The left-hand side of (24)′ is the opportunity cost of stopping, lost flow Θ plus the loss in expected capital gains in project value plus the lost value of information. The right-hand side of (24)′ is the opportunity benefit of stopping, the interest flow from investing the proceeds of stopping. After normalizing by Θ, (24)′ can be solved for $\lambda = \left((1-\tfrac{1}{a_1})(1-\rho c)\right)^{-1}$.

With infinitely repeated options to invest in a new level of productivity the forward NPV is

$Y = \theta/\rho - c\theta - \Theta/\rho + \theta A_1$. The functional form of the option value is unchanged. Now, the r% rule is to invest when[18]

$$\Theta + \alpha\theta(\tfrac{1}{\rho} - c + A_1) + \tfrac{1}{2}\sigma^2 \tfrac{a_1}{\rho}\Theta = \rho\theta(\tfrac{1}{\rho} - c + A_1),$$

or

$$\Theta + \alpha\theta(\tfrac{1}{\rho} - c) - (\rho - \alpha)\theta A_1 + \tfrac{1}{2}\sigma^2 \tfrac{a_1}{\rho}\Theta = \rho\theta(\tfrac{1}{\rho} - c). \qquad (24)''$$

After normalizing by Θ, (24)″ can be solved iteratively for $\lambda < \left((1-\tfrac{1}{a_1})(1-\rho c)\right)^{-1}$.

[17] It can be shown through algebraic manipulation that this is equivalent to equation (24) in Malchow-Møller and Thorsen (2005).

[18] It can be shown through algebraic manipulation that this is equivalent to equation (25) in Malchow-Møller and Thorsen (2005).

The new term $(\rho-\alpha)\theta A_1 > 0$ on the left-hand side of (24)″ reflects a net reduction in pure postponement flow for any θ/Θ, causing investment to happen at a lower value of λ. Since the quasi-option adjustment term is the same in (24)′ and (24)″, this class of repeated options to invest do not reduce the irreversibility of investment as we have defined it, and nor do they reduce the value of information from waiting to invest.

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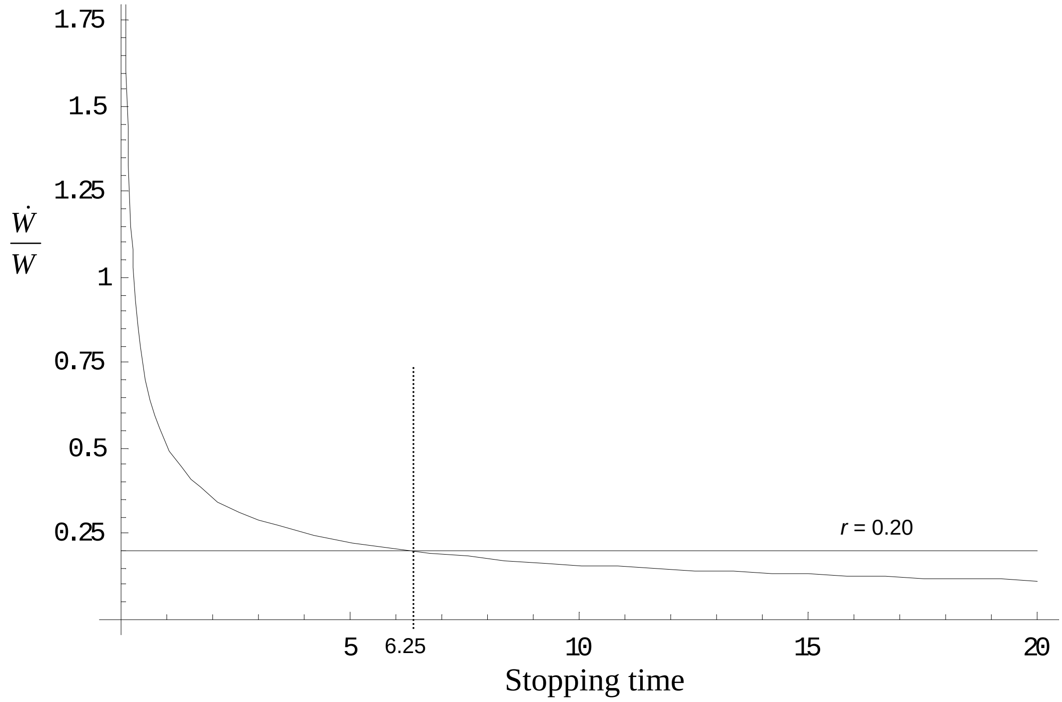

Figure 1: Rate of appreciation of underlying asset, Appendix 1, r = 0.20.

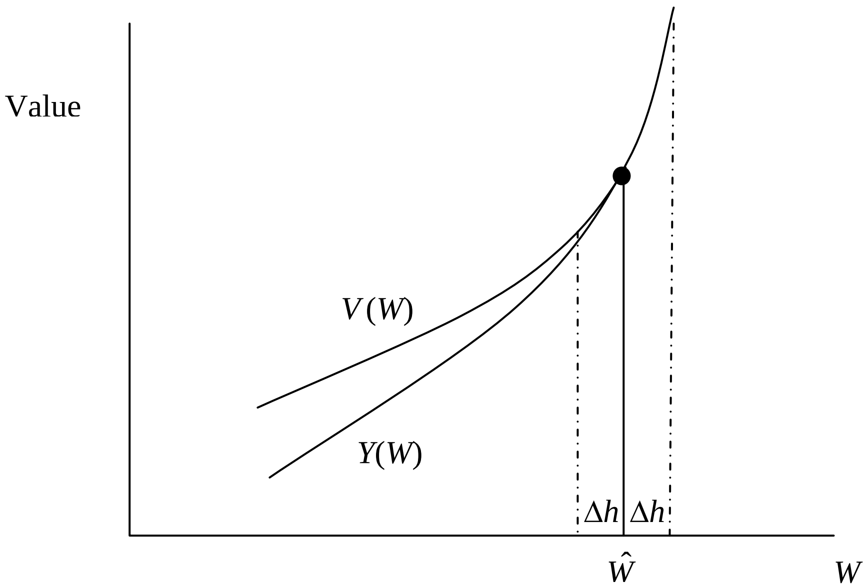

Figure 2: Possible deviations in forward NPV Y and program value V given $W = \hat{W}$.

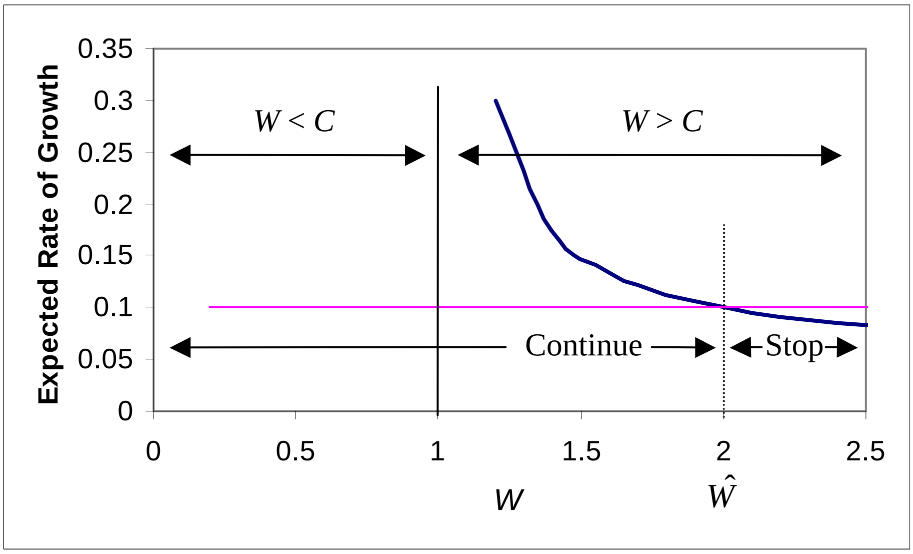

Figure 3: The *r*% rule for geometric Brownian motion, Appendix 2.B, comparing expected rate of growth of the project forward NPV *Y* with the adjusted force of interest $\rho^* = \rho - \alpha = \rho - \frac{1}{2}\sigma^2\beta(\beta - 1) = 0.10$ for $C = 1$, $r = 0.06$, $u = 0.10$, $\rho = 0.14$, $b = 0.05$, and $\sigma = 0.20$.

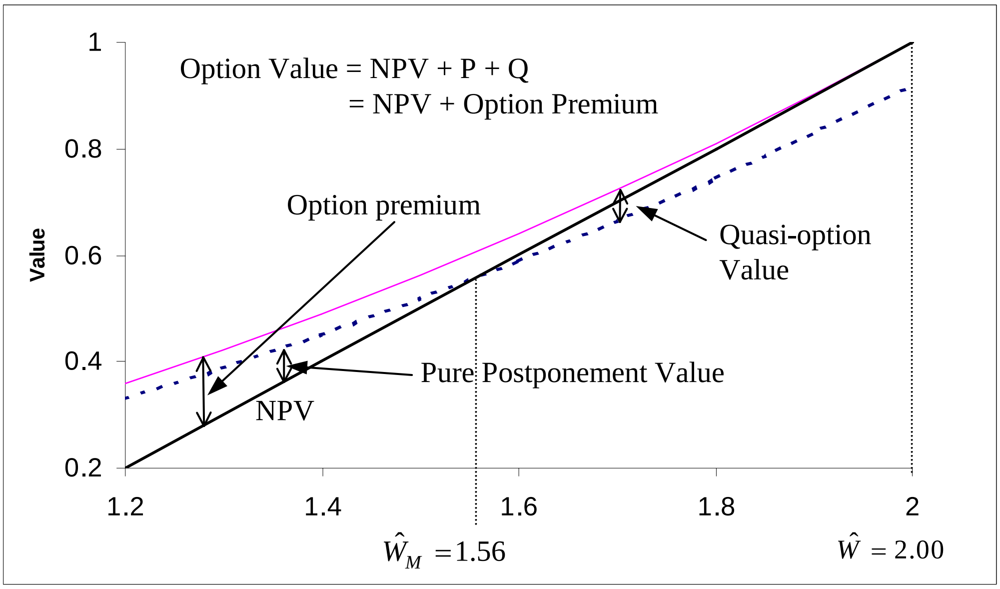

Figure 4: Investment timing under geometric Brownian motion, Appendix 2.B, showing option premium, pure postponement value, and quasi-option value for the parameter values given in Figure 3.

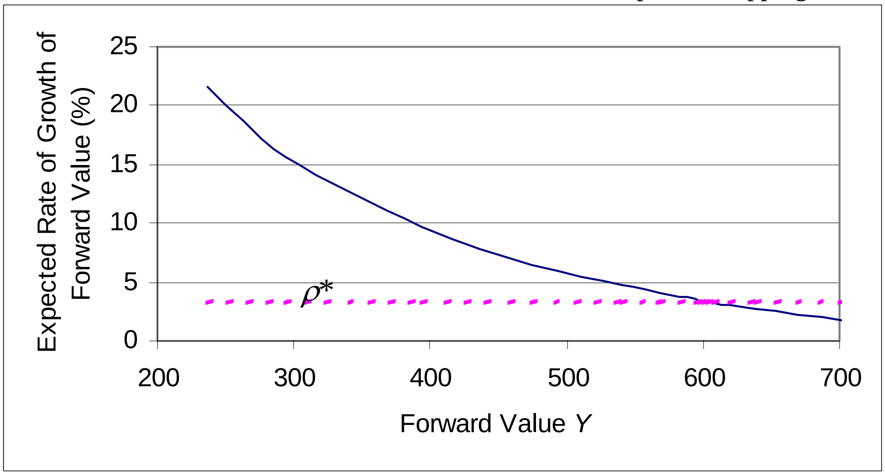

Figure 5: The *r*% rule for the Clarke and Reed oil well abandonment problem. The well is abandoned when the forward NPV of abandoning, *Y*, rises to 608.08, at which point the expected rate of increase in forward NPV falls to an adjusted force of interest, $\rho^* = \rho - \alpha$, of 3.2173%.

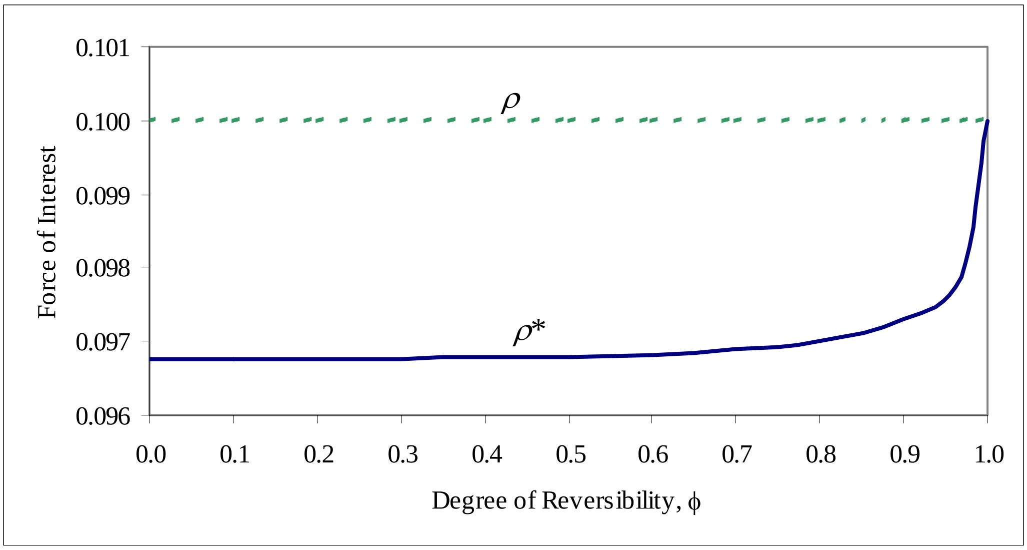

Figure A3.1: Adjusted force of interest, $\rho^* = \rho - \alpha$, for the Brennan and Schwartz model, k_2 = \$1 million, $k_1 = -\phi k_2$, q = 10 million units/yr., a = \$0/unit, f = \$0/yr., π = 0%/yr., κ = 1%/yr., σ^2 = 8%/yr., no taxes, ρ = 10%/yr. The Figure demonstrates that as reversibility becomes complete the quasi-option flow α, the value of information from waiting, goes to zero.